Math 121: Challenge problem

Feel free to come discuss with me any time! Including after the term is over. If you have a solution to all or part of this problem, I'd love to hear it.

1. Let's think about the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

   We've learned that this series sums to 1, since it's geometric with $r = 1/2$ and first term $1/2$. Let's consider the numbers that we can get by adding up only *some* of these terms, and throwing the rest away. Clearly we can get $\frac{1}{2}$, by adding just the first term and throwing the rest away. Similarly, we can get $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc. We can also get $\frac{3}{4}$ by adding the first two terms and throwing the rest away.

   We can add up infinitely many – but still not all – of the terms as well. For instance, we can get $\frac{1}{3}$ like this:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{n-1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

   Let's call the set of all numbers we can obtain in this way $SS(\sum \frac{1}{2^n})$. That stands for the *sub-sums* of the series (I left off the $n = 1$ to ∞ part to simplify the notation). Since the smallest sub-sum we can get is 0 (if we add no terms) and the largest is 1 (if we add all the terms), $SS(\sum \frac{1}{2^n})$ is contained in the interval $[0, 1]$.

   Your first task is to figure out exactly what $SS(\sum \frac{1}{2^n})$ is. [Hint: think about binary expansions – every number between 0 and 1 has a binary expansion of the form $0.d_1 d_2 d_3 \cdots$, where each d_i is 0 or 1 and d_1 denotes the 1/2 place, d_2 the 1/4 place, etc. (If we were in base 10, d_1 would denote the 1/10 place, d_2 the 1/100 place, etc.)

2. Now consider the series

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots .$$

   Compare and contrast $SS(\sum \frac{2}{3^n})$ with $SS(\sum \frac{1}{2^n})$.

3. Let's turn our attention to the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. Find some numbers that are in $SS(\sum \frac{1}{n})$. Does $SS(\sum \frac{1}{n})$ include the interval $[0, 1]$? Is $SS(\sum \frac{1}{n}) = [0, \infty)$?